

Scalar field collapse in three-dimensional AdS spacetime

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We describe results of a numerical calculation of circularly symmetric scalar field collapse in three spacetime dimensions with negative cosmological constant. The procedure uses a double null formulation of the Einstein-scalar equations. We see evidence of black hole formation on first implosion of a scalar pulse if the initial pulse amplitude A is greater than a critical value A_* . Sufficiently near criticality the apparent horizon radius r_{AH} grows with pulse amplitude according to the formula $r_{AH} \sim (A - A_*)^{0.81}$.

I. INTRODUCTION

Dynamical gravitational collapse of matter field configurations continues to be an interesting problem in general relativity. The questions concern whether the long time static or stationary limit of collapse results in a black hole or a naked singularity, the details of the final outcome, and the dependence of results on matter type and symmetries of the system.

Among the simplest such systems is the Einstein-scalar equations with minimally coupled massless scalar field with certain symmetries. Initial work on this system in four spacetime dimensions in spherical symmetry was due to Christodoulou [1], who showed that there exist classes of initial scalar configurations which give regular solutions for all time. Subsequently, numerical work by Goldwirth and Piran [2] established that black hole formation occurs for certain classes of initial data. A few years later a comprehensive study of this system by Choptuik [3] answered several important questions. In particular he showed that regular initial data characterized by parameter p leads to a black hole with mass $M \sim (p - p_*)^\gamma$ where $p_* < p$ is a certain critical value of p and γ is a numerically determined exponent. This work also showed that the solution $p = p_*$ is a naked singularity.

Since this initial work many other systems have been studied and similar behaviour found at the threshold of black hole formation. A semi-analytical understanding of the exponent in the mass formula has been obtained by assuming a critical solution and studying the linearized perturbation equations. For a comprehensive review and references see Ref. [4].

In the present work we describe a numerical simulation of minimally coupled scalar field collapse in three spacetime dimensions with negative cosmological constant. This system is of interest for a number of reasons. (i) A length scale set by the cosmological constant is present unlike the four dimensional Einstein-scalar case, where the only scale (apart from

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the gravitational constant) enters via initial data. (ii) There is a static black hole solution in three dimensions [5] with mass determined in units of the (negative) cosmological constant. In particular, there is no black hole with zero cosmological constant. (iii) The asymptotic spacetime is anti-deSitter (AdS) and not Minkowski, which may permit the possibility of reflecting an outgoing scalar field back to the interior.

Furthermore, on a separate but related note, there is currently interest in asymptotically anti-deSitter metrics from string theory, where the so called AdS/CFT correspondence has been proposed between gravitational phenomena in the bulk of anti-deSitter spacetime and physical effects in certain conformally invariant Yang-Mills theories on its boundary [6]. The essence of the correspondence are certain proposed ‘dictionary’ entries which translate between gravitational and Yang-Mills phenomena. In particular, the correspondence relates low energy gravity processes with high energy Yang-Mills ones, and vice versa. Although this correspondence is for five dimensional AdS, there is an analog of the correspondence relevant to three dimensions. Thus, any critical or other behaviour found in matter collapse in the bulk AdS spacetime may provide hints for a dictionary entry for the AdS/CFT conjecture. As such results of the present work are potentially of interest for string theory.

There has been other work, both analytical and numerical on matter collapse in three dimensions. There are Vaidya-like solutions [8] for collapsing dust, with and without pressure. Analytic results have been found for the collapse of dust shells [9] and point particles [10]. A recent numerical study of this same system has been carried out by Pretorius and Choptuik using a different coordinatization and numerical technique [11]. More recently an exact critical self similar solution has been found [12].

This paper is organized as follows. The next section presents the system of equations and the details of the boundary conditions we use. Section III describes the numerical procedure. The final section describes the results and outline of possible future work.

II. EINSTEIN-SCALAR EQUATIONS

The Einstein-scalar equations with cosmological constant $\Lambda \equiv -1/l^2$ are

$$G_{ab} - \frac{1}{l^2} g_{ab} = 2\pi T_{ab}, \quad (1)$$

with

$$T_{ab} = \partial_a \phi \partial_b \phi - \frac{1}{2} g_{ab} g^{cd} \partial_c \phi \partial_d \phi. \quad (2)$$

These may be rewritten in the form

$$R_{ab} + \frac{2}{l^2} g_{ab} = 2\pi \partial_a \phi \partial_b \phi. \quad (3)$$

The general circularly symmetric metric in three spacetime dimensions in double null coordinates (u, v) may be written as

$$ds^2 = g_{ab} dx^a dx^b = -2l g(u, v) r'(u, v) dudv + l^2 r^2(u, v) d\theta^2, \quad (4)$$

where we are taking g and r to be dimensionless functions, (u, v) and l have dimensions of length, and $f' = \partial f / \partial v$, $\dot{f} = \partial f / \partial u$. In this parametrization, coordinate freedom has been fixed since there are only two metric functions to be determined. Time evolution may be viewed as occurring with respect to the coordinate u .

The vv component of the Einstein equations is the constraint

$$\frac{g'r'}{gr} = 2\pi(\phi')^2, \quad (5)$$

on a fixed u surface. The $\theta\theta$ component is the time (or u) evolution of r

$$l\dot{r}' = -rr'g. \quad (6)$$

Finally, the time evolution of the scalar field is provided by the Klein-Gordon equation

$$(r\phi')' + (r\dot{\phi})' = 0. \quad (7)$$

The remaining components of the Einstein equation are identically satisfied as consequence of these three equations.

If $\phi = 0$ these equations admit a black hole solution first discovered in [5]. In the standard (r, t) coordinates the black hole metric is

$$ds^2 = -\left(\left(\frac{r}{l}\right)^2 - M\right) dt^2 + \left(\left(\frac{r}{l}\right)^2 - M\right)^{-1} dr^2 + r^2 d\theta^2. \quad (8)$$

AdS space is recovered by setting $M = -1$. We can label the outgoing radial null geodesics of the AdS metric by u and the ingoing ones by v . These coordinates are defined by

$$u = t - l \tan^{-1}\left(\frac{r}{l}\right) \quad v = t + l \tan^{-1}\left(\frac{r}{l}\right) \quad (9)$$

Therefore AdS space in double null coordinates with our choice of dimensions and parametrization is given by

$$r(u, v) = \tan\left(\frac{v - u}{2l}\right), \quad g(u, v) = 1. \quad (10)$$

Similarly, the BTZ black hole in the corresponding null coordinates is given by ¹

$$r(u, v) = \sqrt{M} \tanh\left[\frac{\sqrt{M}}{2l}(u - v)\right], \quad g(u, v) = 1. \quad (11)$$

Thus $r = 0$ corresponds to $u = v$. In the following we will use this as one of the boundary conditions for the full set of coupled equations.

Equations (5–7) may be rewritten in a first order initial value form suitable for numerical integration as follows: On a constant u surface, integrating the constraint equation (5) gives

¹Note that this is the BTZ solution for $r < \sqrt{M}$. For $r > \sqrt{M}$ the tanh is replaced by coth.

$$g = k_1(u) \exp \left[2\pi \int_u^v d\tilde{v} \left(\frac{r}{r'} (\phi')^2 \right) \right], \quad (12)$$

where $k_1(u)$ is an arbitrary function of u .

Integrating (6) gives

$$\dot{r} = -\frac{1}{l} \int_u^v d\tilde{v} (g r r') + \frac{k_2(u)}{l}, \quad (13)$$

where $k_2(u)$ is an arbitrary function of u . Note that in both of these equations we are integrating from u to v which is to correspond to integrating outward from $r = 0$.

Finally, using \dot{r} from (13), and the definition

$$h \equiv \frac{2r\phi' + r'\phi}{r'}, \quad (14)$$

the Klein-Gordon equation (7) gives

$$\dot{h} = \frac{r}{l} (h - \phi) \left(g + \frac{k_2(u) - \bar{g}}{2r^2} \right), \quad (15)$$

where

$$\bar{g}(u, v) \equiv \int_u^v d\tilde{v} (g r r'). \quad (16)$$

The definition of h Eqn. (14) gives

$$\phi(u, v) = \frac{1}{2\sqrt{r}} \int_u^v d\tilde{v} \left(\frac{h r'}{\sqrt{r}} \right) + \frac{k_3(u)}{\sqrt{r}}. \quad (17)$$

The function $k_3(u)$ is fixed by noting from Eqn. (14) if $\phi = C$ then $h = C$, where C is a constant. Comparing this with Eqn. (17) gives $k_3(u) = 0$.

The equations (12), (13), and (15) give the required null initial value formulation for the three-dimensional Einstein-scalar system with negative cosmological constant $\Lambda = -1/l^2$. These equations contain two arbitrary functions $k_1(u)$ and $k_2(u)$, which are to be fixed by boundary conditions. With these functions fixed, initial data sets $r(u = 0, v)$ and $h(u = 0, v)$ (derived from ϕ) can be evolved to later times u .

Let us now see how the arbitrary functions in the equations are fixed. The most natural boundary condition is the requirement that the metric is regular at the origin $r = 0$, which is a timelike line. In our formulation $r(u, v)$ is a field variable. Therefore we must first define where $r = 0$ is in order to impose regularity conditions there. As noted above the natural choice motivated by AdS and the BTZ black hole is $r = 0$ at $u = v$, i.e.

$$r(u, u) = 0, \quad (18)$$

which implies

$$\dot{r}(u, u) = -r'(u, u). \quad (19)$$

With this choice for $r = 0$, there are two possible boundary conditions that may be imposed at $r = 0$. Before describing these we set $k_1(u) = 1$ for simplicity.

(i) No conical singularity at $r = 0$. This gives the condition

$$\dot{r}(u, u) = -n^2 g(u, u)/2l. \quad (20)$$

where n is an integer. (See the Appendix.) Using this in Eqn. (13) gives

$$k_2(u) = -\frac{n^2 g(u, u)}{2} = -\frac{n^2}{2}, \quad (21)$$

which is to be used in the evolution equations (13) and (15).

(ii) The uv metric function is unity:

$$2l \, g(u, u) \, r'(u, u) = -2l g(u, u) \, \dot{r}(u, u) = 1. \quad (22)$$

Therefore

$$\dot{r}(u, u) = -\frac{1}{2l g(u, u)}, \quad (23)$$

and eqn. (13) gives

$$k_2(u) = -\frac{1}{2} \quad (24)$$

We shall use $k_2(u) = -1/2$. In summary the first order equations to be integrated numerically are

$$\dot{r} = -\frac{\bar{g}}{l} - \frac{1}{2l} \quad (25)$$

$$\dot{h} = \frac{r}{l} (h - \phi) \left(g - \frac{\bar{g}}{2r^2} - \frac{1}{4r^2} \right) \quad (26)$$

where

$$g = \exp \left[\frac{\pi}{2} \int_u^v d\tilde{v} \left(\frac{r'}{r} (h - \phi)^2 \right) \right] \quad (27)$$

$$\bar{g} = \int_u^v d\tilde{v} \, (g r r') \quad (28)$$

$$\phi = \frac{1}{2\sqrt{r}} \int_u^v d\tilde{v} \left(\frac{h \, r'}{\sqrt{r}} \right). \quad (29)$$

Notice that with $h = \phi = 0$ our equations with k_2 an arbitrary constant reduce to

$$\dot{r}(u, v) = -\frac{r^2 - 2k_2}{2l}, \quad (30)$$

which is to be integrated with the initial condition $r(u, u) = 0$. The solution is

$$r(u, v) = \sqrt{2k_2} \tanh \left[\frac{\sqrt{2k_2}}{2l} (u - v) \right]. \quad (31)$$

Comparing with (11), we see that

$$M = 2k_2. \quad (32)$$

Thus the value $k_2 = -1/2$, which we use for numerical integration, corresponds to $M = -1$, which is 3-dimensional AdS. Notice that although this is the natural case to study in that it is most similar to the spherically symmetric collapse in four dimensions (because $r = 0$ is a regular point initially), the other choices for k_2 corresponding to $M = 0$, $M > 0$ and $-1 < M < 0$ are also interesting.

III. NUMERICAL METHOD

Our numerical scheme is based on the approach used in [2], and its refinements in [7].

The evolution equations (25–26) are converted into set of coupled ordinary differential equations on a grid by the correspondence

$$h(u, v) \rightarrow h_n(u), \quad r(u, v) \rightarrow r_n(u), \quad (33)$$

and similarly for the derived variables $g(u, v)$, $\bar{g}(u, v)$, and $\phi(u, v)$ where $n = 0, \dots, N$ specifies the v grid. The differential equations are therefore

$$\dot{r}_n = -\frac{\bar{g}_n}{l} - \frac{1}{2l} \quad (34)$$

$$\dot{h}_n = \frac{r_n}{l} (h_n - \phi_n) \left(g_n - \frac{\bar{g}_n}{2r_n^2} - \frac{1}{4r_n^2} \right) \quad (35)$$

We take the initial value functions to be

$$r(0, v) = v, \quad (36)$$

with $h(0, v)$ determined by this and the initial scalar field configuration

$$\phi(u = 0, r) = Ar^2 \exp\left(-\frac{r - r_0}{\sigma^2}\right), \quad (37)$$

where A, σ and r_0 are initial data parameters of the scalar field. The initial specification of the other functions is completed by computing the integrals for g_n and \bar{g}_n using Simpson's rule.

The boundary conditions at fixed u are

$$r_k = 0, \quad \bar{g}_k = 0, \quad g_k = 1. \quad (38)$$

where k is the index corresponding to the position of the origin $r = 0$. (In the algorithm used all grid points $0 \leq i \leq k - 1$ correspond to ingoing rays that have reached the origin and are dropped from the grid; see below). These conditions are equivalent to $r(u, u) = 0$, $g|_{r=0} = g(u, u) = 1$ etc., and guarantee regularity of the metric at $r = 0$. Notice that for our initial data, ϕ_k and hence h_k are initially zero, and therefore remain zero at the origin because of Eqn. (35).

Our numerical procedure is similar to that used in Refs. [2,7]. Time evolution with respect to the coordinate u is initiated via a fourth order Runge-Kutta scheme. The constraint integrals are recalculated on each u slice. As evolution proceeds the entries in the r_n array sequentially reach $r = 0$, at which point they are dropped from the grid. Thus the radial grid loses points with evolution. This is similar to the procedure used in [2] and [7]. At each u step, a check is made to see if an apparent horizon has formed. This is done in the standard way by calculating and plotting the function

$$g^{ab}\partial_a r \partial_b r = -\frac{\dot{r}}{lg}. \quad (39)$$

Evolution terminates if either an apparent horizon forms, or the scalar field pulse reflects off $r = 0$. In practice we stop the evolution if $|\partial_u r|$ becomes 10^{-4} .

To improve numerical accuracy near $r = 0$ we follow a procedure similar to [7], where all functions on a constant u surface are expanded in power series in r at $r = 0$, and the first three values of the constraint integrals are calculated using the respective power series. We write

$$h = h_0 + h_1 r \quad (40)$$

from which the following relations are derived:

$$\phi = h_0 + \frac{h_1 r}{3}, \quad g = 1 + \frac{\pi h_1^2 r^2}{9}, \quad \bar{g} = \frac{r^2}{2} + \frac{r^2 \ln g}{4}. \quad (41)$$

In practice, because of our boundary conditions, h_0 turns out to be of order 10^{-5} . The remaining $N - 3$ values of these functions are computed using Simpson's rule for equally spaced points. This substantially improves accuracy near $r = 0$. v -derivatives of functions are calculated using $f'_i = (f_{i+1} - f_{i-1})/2\Delta v$ with end point values determined by linear extrapolation: $f'_1 = 2f'_2 - f'_3$ and $f'_N = 2f'_{N-1} - f'_{N-2}$.

All numerical calculations were performed for $l = 1$ with up to 1400 v -grid points and scalar field parameters $\sigma = 0.3$ and $r_0 = 1.0$. The code is written in C and was executed on a Sun workstation.

Before describing our results in the next section, we comment further on the numerical procedure outlined above.

Firstly, because we use double null coordinates with the number of v grid points decreasing as ingoing null geodesics cross $r = 0$, we are not able to follow a reflected pulse back out toward infinity. This is likely a shortcoming because after first implosion some percentage of the scalar field intensity will be reflected back toward infinity and reach it in finite time (due to the compactness of AdS space). It is then possible that some part of the field would be reflected back toward the origin from the boundary at infinity. This would tend to increase the mass of the black hole. Multiple reflections could occur several times until a static limit is reached. Our procedure is limited to evaluating the mass of the black hole on first implosion of the scalar field pulse. We can assume however that the mass increase due to multiple reflections would be small compared to the mass obtained on first implosion.

Secondly, again due to the loss of points from the grid, we are not able to observe the behaviour of the scalar field very near criticality. However, the calculations performed in [11] indicate that there is continuous self similarity in the near critical regime.²

IV. RESULTS AND COMMENTS

After several simulations for different values of the initial amplitude A of the scalar field, we arrive at the following conclusions. For amplitudes smaller than $A_* \simeq 0.1248$ the field is reflected back in totality from the origin and no black hole is formed. For $A > A_*$ however, we observe black hole formation.

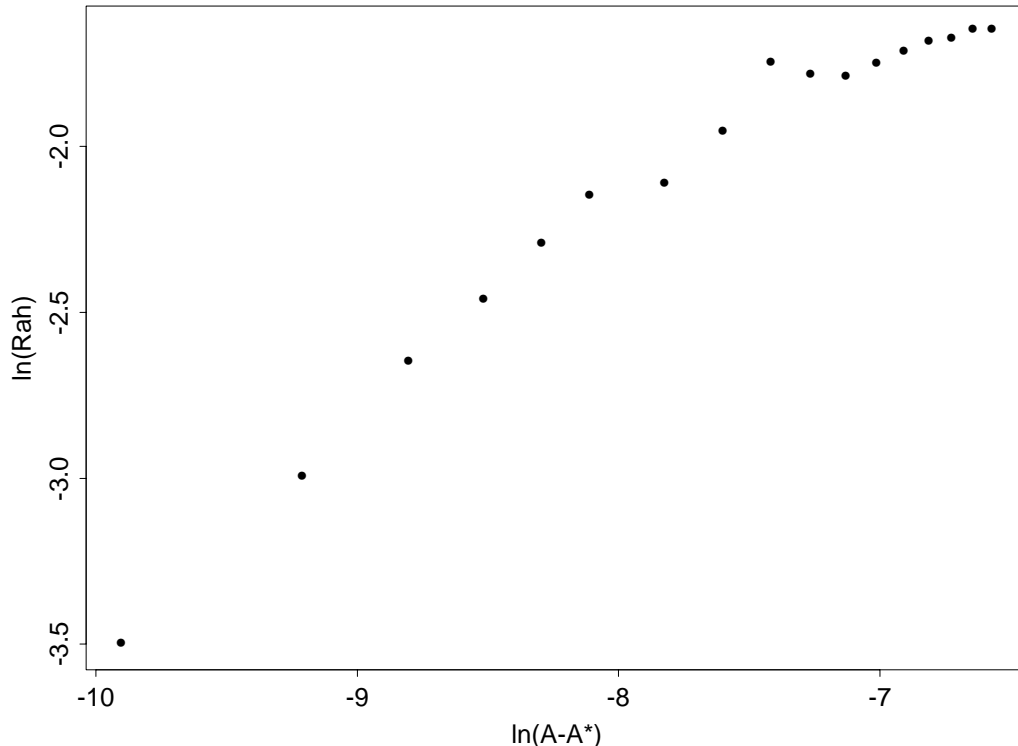


FIG. 1. Logarithmic plot of the apparent horizon radius r_{AH} versus the initial amplitude of the imploding scalar field $(A - A_*)$.

Figure 1 shows a logarithmic plot of the apparent horizon radius r_{AH} versus the initial amplitude A . The plot shows that for A sufficiently close to A_* the behaviour is nearly linear (the first six points). This is the region of interest for us since it appears to exhibit the type of scaling behaviour observed in many other collapse studies [3,4]. We ignore the other part of the plot which shows some oscillatory behaviour and eventually a flattening of

²As in our case, and for similar reasons, this work also focusses on a search for critical and scaling behaviour on first implosion of the scalar pulse.

the curve, because this is farther from criticality. We thus estimate the value of the critical exponent by considering the linear part. This yields the result

$$r_{AH} \sim (A - A_*)^\gamma \quad (42)$$

where $\gamma \simeq 0.81$. Given that black-hole mass is proportional to r_{AH}^2 for the BTZ black-hole, our mass exponent estimate is 1.62. Pretorius and Choptuik [11] used a different method proposed by Garfinkle to estimate the critical exponent. They studied the subcritical ($A < A_*$) scaling behaviour of the maximum value of the scalar curvature R to obtain a value for γ in the range $1.15 - 1.25$.

There are several directions for future study using the double null procedure outlined here. These include computations for other values of the constant k_2 and other initial scalar configurations. In addition, the self similar behaviour of the scalar field near criticality found numerically in [11] and analytically in [12], can likely be reproduced using the method of grid point restoration used in [7].

In connection with the AdS/CFT conjecture, it will be useful to carry out similar calculation for 5-dimensional AdS space. The computational procedure used here requires relatively minor modifications to study that case.

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V. APPENDIX

The condition of no conical singularity at the origin $r = 0$ is derived. The parallel transport of a vector A^a on a closed loop around the origin should be such that this vector comes back to itself after a complete turn. The condition for parallel transport of A^a along a curve tangential to t^a is

$$t^a \nabla_a A^b = 0. \quad (43)$$

Moving on a curve of constant u and v , only θ varies from 0 to 2π , i.e. $t^a = (\partial/\partial\theta)^a$ and the above equation becomes

$$\nabla_\theta A^b = \partial_\theta A^b + \Gamma_{\theta c}^b A^c = 0. \quad (44)$$

The relevant non-zero Γ are

$$\Gamma_{\theta\theta}^u = \frac{r}{g} \quad \Gamma_{\theta\theta}^v = \frac{r\dot{r}}{gr'} \quad (45)$$

$$\Gamma_{\theta u}^\theta = \frac{\dot{r}}{r} \quad \Gamma_{\theta v}^\theta = \frac{r'}{r}. \quad (46)$$

The parallel transport equation is then

$$\partial_\theta \begin{bmatrix} A^u \\ A^v \\ A^\theta \end{bmatrix} = \begin{bmatrix} 0 & 0 & -\frac{r}{g} \\ 0 & 0 & -\frac{r\dot{r}}{gr'} \\ -\frac{\dot{r}}{r} & -\frac{r'}{r} & 0 \end{bmatrix} \begin{bmatrix} A^u \\ A^v \\ A^\theta \end{bmatrix}. \quad (47)$$

Diagonalizing the matrix on the right we get 3 eigenvalues $\lambda = 0, \pm\sqrt{\frac{2\dot{r}}{g}}$. In the eigenvectors basis we would thus have the following differential equation to solve for each component

$$\partial_\theta A^\alpha = \lambda^{(\alpha)} A^\alpha. \quad (48)$$

This is solved by $A^\alpha(\theta) = A^\alpha(0) \exp(\lambda^{(\alpha)}\theta)$. Requiring that A^α be unchanged after a complete non-trivial turn around the origin requires

$$A^\alpha(2\pi) = A^\alpha(0) \Rightarrow A^\alpha(0) \exp(2\pi\lambda^{(\alpha)}) = A^\alpha(0) \Rightarrow \lambda^{(\alpha)} = ni, \quad (49)$$

where $n \in \mathbb{Z}$. The simplest non-trivial case is $\lambda^{(\alpha)} = i$, and this becomes a boundary condition if we look at the case where α is the third eigenvalue :

$$\sqrt{\frac{2\dot{r}}{g}} = i \quad \text{at} \quad r = 0. \quad (50)$$

This first boundary condition is thus

$$\dot{r} = -\frac{g}{2} \quad \text{at} \quad r = 0. \quad (51)$$

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